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CRACKS WITH SMOOTHLY CLOSING EDGES UNDER PLANE DEFORMATION*

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Asymptotic methods are used to compute the stress intensity coefficients at the tips of a thin cut with smoothly closing edges, and the asymptotic form of the potential energy is determined.

Amongst various mathematical models representing real cracks, the "crack-cut" model is of special interest, since it requires the simplest mathematical methods in its study. However, the model does not reflect some of the properties of actual cracks, in particular the crack does not respond, within its framework, to loading in the direction of the cut. The case of a thin cut with smoothly closing edges ensures good agreement with reality, while retaining the simplicity of the cut**. The present paper deals chiefly with the explanation of how sensitive such cracks are to the loading along the cut.

1. Formulation of the problem and preliminary discussion.

Let Ω be a region in the $\mathbf{R}^2$ plane, containing a segment $M=\{x\colon x_2=0,\ |x_1|\leqslant a\}$, $h_\pm(x_1)=ah_\pm^\circ(x_1)$; $h_\pm^\circ$ are functions smooth on $[-a,a]$ such that $h_\pm^\circ(a)=h_\pm^\circ(-a)=h_\pm^{\circ\prime}(a)=h_\pm^{\circ\prime}(-a)=0$. We introduce the regions $G_\varepsilon=\{x\colon |x_1|<a,\ -\varepsilon h_-(x_1)<x_2<\varepsilon h_+(x_1)\}$, $\Omega_\varepsilon=\Omega\setminus G_\varepsilon$ depending on the small positive dimensionless parameter ε. Let us consider the plane problem of the theory of elasticity

$$\mu\Delta\mathbf{u}(x,\ \varepsilon)+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}(x,\ \varepsilon)=0 \text{ in } \Omega_\varepsilon \tag{1.1}$$

$$\sigma^{(n)}(\mathbf{u};\ x,\ \varepsilon)=\mathbf{p}^\pm(x_1,\ \varepsilon) \quad \text{on } \gamma_\varepsilon^\pm=\{x\in\partial G_\varepsilon\colon \pm x_2>0\} \tag{1.2}$$

$$\sigma^{(n)}(\mathbf{u};\ x,\ \varepsilon)=\mathbf{p}(x) \text{ on } \partial\Omega \tag{1.3}$$

where $\mathbf{p}, \mathbf{p}^\pm$ are smooth loads, $\mathbf{u}$ is the displacement vector, σ is the stress tensor, λ, μ are the Lamé constants and $n=(n_1,\ n_2)$ is the external normal for the region Ω_ε.

The problem concerning the effect of the transverse size of the crack engaged the attention of workers for a long time. It was shown in /2, 3/ (see also /1/) that it is best to use asymptotic methods with such problems, and the stress-deformation state near a thin cut with a smooth boundary was investigated. In this case a boundary layer appears near the tip of the cut, described with help of a solution of the problem on the outside of a parabola. The mathematical treatment of these problems was continued in /4/.

The present paper also uses expansions in series in the small parameter ε. However, unlike the previous papers, the restrictions imposed here on the geometry of the boundary near the crack tips are such that the boundary layer gives rise to an insiginficant (uniformly exponentially small in ε) perturbation (see /5/). A power series in ε appears away from the crack tips, and therefore the exponentially small terms should be neglected. Thus, below we concentrate our attention on studying the effect which the curving of the middle part of the crack edges has on the stress-deformation state.

We will assume that the external load is selfequilibrated, i.e.

$$\int_{\partial\Omega}\mathbf{p}(x)\,ds=-\sum_\pm\int_{\gamma_\varepsilon^\pm}\mathbf{p}^\pm(\varepsilon,\ x_1)\,ds$$

Since the arc length element ds is equal to $\pm(1+\varepsilon^2h_\pm'(x_1)^2)^{1/2}dx_1$ on $\gamma_\varepsilon^\pm$, the last equation takes the form

$$\int_{\partial\Omega}\mathbf{p}(x)\,ds=-\sum_\pm\int_{-a}^{a}\mathbf{p}^\pm(\varepsilon,\ x_1)(1+\varepsilon^2h_\pm'(x_1)^2)^{1/2}\,dx_1$$

Therefore we shall assume that in (1.2) $\mathbf{p}^\pm(\varepsilon,\ x_1)=(1+\varepsilon^2h_\pm'(x_1)^2)^{-1/2}\mathbf{q}^\pm(x_1)$. It is clear that

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**The need to investigate such cases was pointed out in /1/.

$$\mathbf{p}^{\pm}(\varepsilon, x_1) = \mathbf{q}^{\pm}(x_1) - {}^1/_2\varepsilon^2 h_{\pm}'(x_1)^2\mathbf{q}^{\pm}(x_1) + O(\varepsilon^4)$$

Let us write $\varepsilon = 0$. Then the region Ω_ε will be transformed into the region $\Omega_0 = \Omega \setminus M$ with a crack M, and the boundary value problem (1.1)-(1.3) will become

$$\mu\Delta\mathbf{u}^\circ(x) + (\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{u}^\circ(x) = 0 \quad \text{in} \quad \Omega_0 \tag{1.4}$$

$$\sigma_{2j}(\mathbf{u}^\circ, x_1, \pm 0) = \mp q_j^{\pm},\ j = 1, 2 \quad \text{when} \quad |x_1| < a \tag{1.5}$$

$$\sigma^{(n)}(\mathbf{u}^{(0)}; x) = \mathbf{p}(x) \text{ on } \partial\Omega \tag{1.6}$$

We know (see e.g., /6, 7/) that the solution $\mathbf{u}^\circ$ of problem (1.4)-(1.6) admits, near the tips $O_{\pm} = (\pm a, 0)$ of the crack M, of the asymptotic representations

$$\begin{aligned}
u_r^{\pm}(r, \theta_{\pm}) = {} & C_1^{\pm}\cos\theta_{\pm} + C_2^{\pm}\sin\theta_{\pm} + \\
& \frac{1}{4\mu}\left(\frac{r_{\pm}}{2\pi}\right)^{1/2}\left\{\left[(2\varkappa - 1)\cos\frac{\theta_{\pm}}{2} - \cos\frac{3\theta_{\pm}}{2}\right]K^{\pm}_{(1, 0)} - \right.\\
& \left.\left[(2\varkappa - 1)\sin\frac{\theta_{\pm}}{2} - 3\sin\frac{3\theta_{\pm}}{2}\right]K^{\pm}_{(2, 0)}\right\} + O(r) \\
u_\theta^{\pm}(r, \theta_{\pm}) = {} & -C_1^{\pm}\sin\theta_{\pm} + C_2^{\pm}\cos\theta_{\pm} + \\
& \frac{1}{4\mu}\left(\frac{r_{\pm}}{2\pi}\right)^{1/2}\left\{\left[-(2\varkappa + 1)\sin\frac{\theta_{\pm}}{2} + \sin\frac{3\theta_{\pm}}{2}\right]K^{\pm}_{(1, 0)} - \right.\\
& \left.\left[(2\varkappa + 1)\cos\frac{\theta_{\pm}}{2} - 3\cos\frac{3\theta_{\pm}}{2}\right]K^{\pm}_{(2, 0)}\right\} + O(r); \quad \varkappa = \frac{\lambda + 3\mu}{\lambda + \mu}
\end{aligned} \tag{1.7}$$

($(r_{\pm}, \theta_{\pm})$ are polar coordinates with centres $O_{\pm}$, and polar axes directed along the segment M, $C_j^{\pm}$ are rigid displacements of the points $O_{\pm}$, $K^{\pm}_{(j,0)}$ are the stress intensity factors). The results of /6, 8, 9/ imply that the coefficients $K^{\pm}_{(j, 0)}$ are given by the formulas

$$K^{\pm}_{(j, 0)} = \frac{1}{2\sqrt{\pi a}}\left\{\int_{\partial\Omega}\mathbf{p}(x)\,\zeta^{(j, \pm)}(x)\,ds + \sum_{\pm}\int_{-a}^{a}\mathbf{q}^{\pm}(x_1)\,\zeta^{(j, \pm)}(x_1, \pm 0)\,dx_1\right\} \tag{1.8}$$

where $\zeta^{(j, \pm)}$ are the non-energetic solutions of the homogeneous $(\mathbf{q}^{\pm} \equiv 0,\ \mathbf{p} \equiv 0)$ problem (1.4)-(1.6) bounded everywhere in $\overline{\Omega}_0 \setminus \{O_{\pm}\}$, with the following asymptotic form near $O_{\pm}$:

$$\begin{aligned}
\zeta^{(1, \pm)}(r, \theta) = {} & (2\pi r_{\pm})^{-1/2}[2(1 + \varkappa)]^{-1}\left((2\varkappa + 1)\cos\frac{3\theta_{\pm}}{2} - \right.\\
& \left. 3\cos\frac{\theta_{\pm}}{2},\ \ 3\sin\frac{\theta_{\pm}}{2} - (2\varkappa - 1)\sin\frac{3\theta_{\pm}}{2}\right) + O(1)
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
\zeta^{(2, \pm)}(r, \theta) = {} & (2\pi r_{\pm})^{-1/2}[2(1 + \varkappa)]^{-1}\left((2\varkappa + 1)\sin\frac{3\theta_{\pm}}{2} - \right.\\
& \left. \sin\frac{\theta_{\pm}}{2},\ \ (2\varkappa - 1)\cos\frac{3\theta_{\pm}}{2} - \cos\frac{\theta_{\pm}}{2}\right) + O(1)
\end{aligned} \tag{1.10}$$

2. Asymptotic form of the intensity factors. Since the functions $h_{\pm}$ are smooth and vanish, together with their derivatives, at the ends of the segment $[-a, a]$, it follows that the solution of problem (1.1)-(1.3) admits, near the points $O_{\pm}$, of the representation (1.7) (see /7/). Let us find the asymptotic form of the coefficients $K_j^{\pm}(\varepsilon)$ and $\varepsilon \to 0$.

Following /10-12/, we shall seek the asymptotic expansion of the solution of the boundary value problem (1.1)-(1.3) in the form

$$\mathbf{u}(x, \varepsilon) \sim \sum_{k=0}^{\infty}\varepsilon^k\mathbf{u}^{(k)}(x) \tag{2.1}$$

where $\mathbf{u}^{(k)}$ are solutions of problems of the form (1.4)-(1.6) in the region Ω_0 with certain right-hand sides. We note that by virtue of the constraints imposed on the function $h_{\pm}$ problem (1.1)-(1.3) must be regarded as a problem in a region with a regularly perturbed boundary.

Let us consider the upper and lower edge $\gamma_\varepsilon^{\pm}$ of the slit G_ε. The unit vector of the internal normal (with respect to G_ε) to $\gamma_\varepsilon^{\pm}$, has the form

$$\mathbf{n}(x_1, \varepsilon) = (1 + \varepsilon^2 h_{\pm}'(x_1)^2)^{-1/2}(\varepsilon h_{\pm}'(x_1), \mp 1)$$

Therefore

$$n_1(x_1, \varepsilon) = \varepsilon h_{\pm}'(x_1) + O(\varepsilon^3),\ n_2(x_1, \varepsilon) = \mp(1 - {}^1/_2\varepsilon^2 h_{\pm}'(x_1)^2) + O(\varepsilon^4)$$

and hence we have, when $x_2 = \pm\varepsilon h_{\pm}(x_1)$

$$\sigma_j^{(n)}(u;\ x) = n_1(x_1,\ \varepsilon)\,\sigma_{1j}(u;\ x) + n_2(x_1,\ \varepsilon)\,\sigma_{2j}(u;\ x) = \\ \mp\sigma_{2j}(u;\ x) + \varepsilon h_\pm'(x_1)\,\sigma_{1j}(u;\ x) \pm {}^1/_2\varepsilon^2 h_\pm(x_1)^2\sigma_{2j}(u;\ x) + O(\varepsilon^3) \tag{2.2}$$

Applying Maclaurin's formula

$$\sigma_{ij}(u;\ x_1,\ \pm\varepsilon h_\pm(x_1)) = \sigma_{ij}(u;\ x_1,\ \pm 0) \pm \varepsilon h_\pm(x_1) \times \\ \sigma_{ij,2}(u; x_1,\ \pm 0) + {}^1/_2\varepsilon^2 h_\pm(x_1)^2\sigma_{ij,22}(u;\ x_1,\ \pm 0) + O(\varepsilon^3)$$

(the subscript k following the comma denotes differentiation with respect to x_k), we reduce the right-hand side of relation (2.2) to the form

$$\sigma_j^{(n)}(u;\ x_1,\ \pm\varepsilon h_\pm(x_1)) = \mp\sigma_{2j}(u;\ x_1,\ \pm 0) + \\ \varepsilon\,\{h_\pm'(x_1)\,\sigma_{1j}(u;\ x_1,\ \pm 0) - h_\pm(x_1)\,\sigma_{2j,2}(u;\ x_1,\ \pm 0)\} \mp \\ {}^1/_2\varepsilon^2\,\{h_\pm(x_1)^2\sigma_{2j,22}(u;\ x_1,\ \pm 0) - 2h_\pm(x_1)\,h_\pm'(x_1) \times \\ \sigma_{1j,2}(u;\ x_1,\ \pm 0) - h_\pm'(x_1)^2\,\sigma_{2j}(u;\ x_1,\ \pm 0)\} + O(\varepsilon^3) \tag{2.3}$$

Combining expansions (2.1) of the solution of problem (1.1)-(1.3) with formulas (2.3), we find that the vector $\mathbf{u}^{(0)}$ is a solution of problem (1.4)-(1.6), and the vector $\mathbf{u}^{(1)}$ satisfies Eqs.(1.4) and boundary conditions

$$\sigma^{(n)}(u^{(1)};\ x) = 0 \text{ on } \partial\Omega \tag{2.4}$$

$$\sigma_{2j}(u^{(1)},\ x_1,\ \pm 0) = \pm h_\pm'(x_1)\,\sigma_{1j}(u^{(0)};\ x_1,\ \pm 0) \mp \\ h_\pm(x_1)\,\sigma_{2j,2}(u^{(0)};\ x_1,\ \pm 0) \text{ when } |x_1| < a \tag{2.5}$$

Eqs.(2.5) can be transformed by virtue of the equations of equilibrium as follows:

$$\sigma_{2j}(u^{(1)};\ x_1,\ \pm 0) = \pm(h_\pm\sigma_{1j,1}(u^{(0)};\ x_1,\ \pm 0) + \\ h_\pm'(x_1)\,\sigma_{1j}(u^{(0)};\ x_1,\ \pm 0)) = \\ \pm\frac{\partial}{\partial x_1}(h_\pm(x_1)\,\sigma_{1j}(u^{(0)};\ x_1,\ \pm 0)) \text{ when } |x_1| < a \tag{2.6}$$

Since $h_\pm(x_1) = O(r_\pm^2)$ as $r_\pm \to 0$, we find, according to (1.7), that the right-hand side of (2.6) is of the order of $O(r_\pm^{1/2})$ as $r_\pm \to 0$. This means (see /7/) that the displacement field $\mathbf{u}^{(1)}$ satisfying (2.4) and (2.6) admits of a representation of the form (1.7) whose coefficients will be denoted by $K^\pm_{(j,1)}$. We also obtain from formulas (1.8) the following expression for the stress intensity factors:

$$K^+_{(j,1)} = -\frac{1}{2\sqrt{\pi a}} \times \\ \sum_\pm \int_{-a}^{a} \frac{\partial}{\partial x_1}(h_\pm(x_1)\,\sigma^{(1)}(u^{(0)};\ x_1,\ \pm 0))\,\zeta^{(+,j)}(x_1,\ \pm 0)\,dx_1 = \\ \frac{1}{2\sqrt{\pi a}} \sum_\pm \int_{-a}^{a} h_\pm(x_1)\,\sigma^{(1)}(u^{(0)};\ x_1,\ \pm 0)\,\frac{\partial\zeta^{(+,j)}}{\partial x_1}(x_1,\ \pm 0)\,dx_1 \\ \sigma^{(j)} = (\sigma_{1j},\ \sigma_{2j})$$

An analogous formula holds for $K^-_{(j,1)}$.

Let us determine the next term of the asymptotic expression. Taking into account in (2.1) and (2.3) terms of the order of $O(\varepsilon^2)$ (which include the term $\varepsilon^2 h_\pm'(x_1)^2\mathbf{q}^\pm(x_1)/2$) in the representation of the right-hand sides $\mathbf{p}^\pm$ of the boundary conditions (1.2), we find that $\mathbf{u}^{(2)}$ should be subject to Eqs.(1.4) with boundary conditions (2.4), and

$$\sigma_{2j}(u^{(2)};\ x_1,\ \pm 0) = \frac{\partial}{\partial x_1}\Big(\frac{1}{2}h_\pm(x_1)^2\,\sigma_{1j,2}(u^{(0)};\ x_1,\ \pm 0) \pm \\ h_\pm(x_1)\,\sigma_{1j}(u^{(1)};\ x_1,\ \pm 0)\Big) + {}^1/_2\,h_\pm'(x_1)^2\,\sigma_{2j}(u^{(0)};\ x_1,\ \pm 0) \pm \\ {}^1/_2\,h_\pm'(x_1)^2\,q_j^\pm(x_1) \text{ when } |x_1| < a \tag{2.7}$$

The right-hand side of Eq.(2.7) is of the order of $O(r_\pm^{1/2})$ as $r_\pm \to 0$, and formulas (1.7) with coefficients $K^\pm_{(j,2)}$ hold for the displacement field $\mathbf{u}^{(2)}$.

The process of constructing the coefficients of the asymptotic expansion can be continued. The resulting representation of the solution of problem (1.1)-(1.3) is justified using standard methods (see e.g. /10-12/). As the result we have the following asymptotic formulas for the stress intensity coefficients:

$$K_j^{\pm}(\varepsilon) = K^{\pm}_{(j,0)} + \varepsilon K^{\pm}_{(j,1)} + \varepsilon^2 K^{\pm}_{(j,2)} + O(\varepsilon^3) \tag{2.8}$$

3. Asymptotic form of the potential energy. Let us consider the potential energy functional

$$\Pi(\varepsilon) = -\frac{1}{2}\int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}(\varepsilon, x)\,ds - \frac{1}{2}\sum_{\pm}\int_{\gamma_\varepsilon^{\pm}} \mathbf{p}^{\pm}(\varepsilon, x_1)\,\mathbf{u}(\varepsilon, x)\,ds \tag{3.1}$$

corresponding to problem (1.1)-(1.3), and obtain its asymptotic form as $\varepsilon \to 0$. According to the expression for the load $\mathbf{p}^{\pm}$ given in Sect.1, we have

$$\Pi(\varepsilon) = -\frac{1}{2}\int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}(\varepsilon, x)\,ds - \frac{1}{2}\sum_{\pm}\int_{-a}^{a} \mathbf{q}^{\pm}(x_1)\,\mathbf{u}(\varepsilon, x_1, \pm\varepsilon h_{\pm}(x_1))\,dx_1 \tag{3.2}$$

The following expression for the displacements holds by virtue of the asymptotic formula (2.1):

$$\int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}(\varepsilon, x)\,ds = \sum_{j=0}^{2} \varepsilon^j \int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}^{(j)}(x)\,ds + O(\varepsilon^3)$$

Since $\mathbf{p}(x) = \boldsymbol{\sigma}^{(n)}(u^{(0)}; x)$, $\boldsymbol{\sigma}^{(n)}(u^{(j)}, x) = 0$ when $j = 1, 2$ on $\partial\Omega$, we use the Betti formula to obtain the relations

$$\int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}^{(j)}(x)\,ds = \int_{\partial\Omega} \{\boldsymbol{\sigma}^{(n)}(u^{(0)}; x)\,\mathbf{u}^{(j)}(x) - \boldsymbol{\sigma}^{(n)}(u^{(j)}; x)\,\mathbf{u}^{(0)}(x)\}\,ds = \tag{3.3}$$

$$\sum_{\pm} \pm \int_{-a}^{a} \{\boldsymbol{\sigma}^{(2)}(u^{(0)}; x_1, \pm 0)\,\mathbf{u}^{(j)}(x_1, \pm 0) - \boldsymbol{\sigma}^{(2)}(u^{(j)}; x_1, \pm 0)\,\mathbf{u}^{(0)}(x_1, \pm 0)\}\,dx_1, \quad j = 1, 2$$

Further, for the integrals in $M_{\pm}$ from the right in (3.2) we have

$$\int_{-a}^{a} \mathbf{q}^{\pm}(x_1)\,\mathbf{u}(\varepsilon, x_1, \pm\varepsilon h_{\pm}(x_1))\,dx_1 = \tag{3.4}$$

$$\int_{-a}^{a} \mathbf{q}^{\pm}(x_1)\Big\{\mathbf{u}^{(0)}(x_1, \pm 0) + \varepsilon\Big[\mathbf{u}^{(1)}(x_1, \pm 0) \pm h_{\pm}(x_1)\frac{\partial \mathbf{u}^{(0)}}{\partial x_2}(x_1, \pm 0)\Big] + \varepsilon^2\Big[\mathbf{u}^{(2)}(x_1, \pm 0) \pm h_{\pm}(x_1)\frac{\partial \mathbf{u}^{(1)}}{\partial x_2}(x_1, \pm 0) + \frac{1}{2}h_{\pm}(x_1)^2\frac{\partial^2 \mathbf{u}^{(0)}}{\partial x_2^2}(x_1, \pm 0)\Big]\Big\}\,dx_1 + O(\varepsilon^3)$$

Combining expressions (3.3) and (3.4) we obtain the following formula from (3.2):

$$\Pi(\varepsilon) = \Pi_0 + \varepsilon\Pi_1 + \varepsilon^2\Pi_2 + O(\varepsilon^3) \tag{3.5}$$

$$\Pi_0 = -\frac{1}{2}\int_{\partial\Omega} \mathbf{p}(x)\,\mathbf{u}^{(0)}(x)\,ds - \frac{1}{2}\sum_{\pm}\int_{-a}^{a} \mathbf{q}^{\pm}(x_1)\,\mathbf{u}^{(0)}(x_1, \pm 0)\,dx_1$$

where Π_0 is the potential energy corresponding to problem (1.4)-(1.6). Collecting in the right-hand sides of (3.3) and (3.4) the coefficients of ε, we obtain

$$\Pi_1 = -\frac{1}{2}\sum_{\pm}\int_{-a}^{a} \Big\{\mp \boldsymbol{\sigma}^{(2)}(u^{(1)}; x_1, \pm 0)\,\mathbf{u}^{(0)}(x_1, \pm 0) \pm \mathbf{q}^{\pm}(x_1)\,h_{\pm}(x_1)\frac{\partial \mathbf{u}^{(0)}}{\partial x_2}(x_1, \pm 0)\Big\}\,dx_1 \tag{3.6}$$

Let us transform expression (3.6) using relations (2.6) and (1.5). We have

$$\Pi_1 = \frac{1}{2}\sum_{\pm}\int_{-a}^{a} \Big[\frac{\partial}{\partial x_1}(h_{\pm}(x_1)\,\boldsymbol{\sigma}^{(1)}(u^{(0)}; x_1, \pm 0)\,\mathbf{u}^{(0)}(x_1, \pm 0) + h_{\pm}(x_1)\,\boldsymbol{\sigma}^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(0)}}{\partial x_2}(x_1, \pm 0)\Big]\,dx_1 = \tag{3.7}$$

$$-\frac{1}{2}\sum_{\pm}\int_{-a}^{a} h_{\pm}(x_1)\Big\{\boldsymbol{\sigma}^{(1)}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(0)}}{\partial x_1}(x_1, \pm 0) - \boldsymbol{\sigma}^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(0)}}{\partial x_2}(x_1, \pm 0)\Big\}\,dx_1$$

Let us find the difference $R_{\pm}(u^{(0)}; x_1)$ appearing in (3.7) within the curly brackets. We have

$$R = (2\mu u_{1,1}^{(0)} + \lambda(u_{1,1}^{(0)} + u_{2,2}^{(0)}))u_{1,1}^{(0)} + \mu(u_{1,2}^{(0)} + u_{2,1}^{(0)})u_{2,1}^{(0)} - \mu(u_{1,2}^{(0)} + u_{2,1}^{(0)})u_{1,2}^{(0)} - (2\mu u_{2,2}^{(0)} + \lambda(u_{1,1}^{(0)} + u_{2,2}^{(0)}))u_{2,2}^{(0)} = (2\mu + \lambda)(u_{1,1}^{(0)2} - u_{2,2}^{(0)2}) + \mu(u_{2,1}^{(0)2} - u_{1,2}^{(0)2})$$

Thus

$$\Pi_1 = -\frac{1}{2}\sum_{\pm}\int_{-a}^{a} h_{\pm}(x_1)\{(2\mu+\lambda)[u_{1,1}^{(0)}(x_1, \pm 0)^2 - u_{2,2}^{(0)}(x_1, \pm 0)^2] + \mu[u_{2,1}^{(0)}(x_1, \pm 0)^2 - u_{1,2}^{(0)}(x_1, \pm 0)^2]\}\,dx_1 \tag{3.8}$$

Similarly, collecting in (3.3) and (3.4) the coefficients of ε^2 and taking into account (1.5) and (2.7), we obtain

$$\Pi_2 = -\frac{1}{2}\sum_{\pm}\int_{-a}^{a}\Big\{\mp \sigma^{(2)}(u^{(2)}; x_1, \pm 0)\,\mathbf{u}^{(0)}(x_1, \pm 0) \pm h_{\pm}(x_1)\,\mathbf{q}^{\pm}(x_1)\frac{\partial \mathbf{u}^{(1)}}{\partial x_2}(x_1, \pm 0) + \frac{1}{2}\mathbf{q}^{\pm}(x_1)\,h_{\pm}(x_1)^2\frac{\partial^2 \mathbf{u}^{(0)}}{\partial x_2^2}(x_1, \pm 0)\Big\}dx_1 =$$
$$\sum_{\pm}\int_{-a}^{a}\Big\{\mp\frac{\partial}{\partial x_1}(h_{\pm}(x_1)^2\frac{1}{2}\frac{\partial \sigma^{(1)}}{\partial x_2}(u^{(0)}; x_1, \pm 0) + h_{\pm}(x_1)\,\sigma^{(1)}(u^{(1)}; x_1, \pm 0))\,\mathbf{u}^{(0)}(x_1, \pm 0) - h_{\pm}(x_1)\,\sigma^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(1)}}{\partial x_2}(x_1, \pm 0) \mp \frac{1}{2}h_{\pm}(x_1)^2\,\sigma^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial^2 \mathbf{u}^{(0)}}{\partial x_2^2}(x_1, \pm 0)\Big\}dx_1$$

or, after integrating by parts,

$$\Pi_2 = -\frac{1}{2}\sum_{\pm}\int_{-a}^{a}\Big\{h_{\pm}(x_1)\Big[\sigma^{(1)}(u^{(1)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(0)}}{\partial x_1}(x_1, \pm 0) - \sigma^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(1)}}{\partial x_2}(x_1, \pm 0) \pm \frac{1}{2}h_{\pm}(x_1)^2\Big[\frac{\partial \sigma^{(1)}}{\partial x_2}(u^{(0)}; x_1, \pm 0)\frac{\partial \mathbf{u}^{(0)}}{\partial x_1}(x_1, \pm 0) - \sigma^{(2)}(u^{(0)}; x_1, \pm 0)\frac{\partial^2 \mathbf{u}^{(0)}}{\partial x_2^2}(x_1, \pm 0)\Big]\Big\}dx_1$$

Finally, let us consider expressions (3.7) in the case of a slit G_ε with free edges, i.e. when $\mathbf{q}^{\pm} = 0$. Then we obtain the following relations for the boundary conditions (1.5):

$$u_{1,2}^{(0)}(x_1, \pm 0) = -u_{2,1}^{(0)}(x_1, \pm 0)$$
$$u_{2,2}^{(0)}(x_1, \pm 0) = -\lambda(2\mu + \lambda)^{-1}u_{1,1}^{(0)}(x_1, \pm 0)$$

Therefore

$$\Pi_1 = -2\mu\frac{\mu+\lambda}{2\mu+\lambda}\sum_{\pm}\int_{-a}^{a} h_{\pm}(x_1)\,u_{1,1}^{(0)}(x_1, \pm 0)^2\,dx_1$$

and we obtain, in accordance with (3.5), the formula

$$\Pi(\varepsilon) = \Pi_0 - \frac{\varepsilon E}{2(1-\nu^2)}\sum_{\pm}\int_{-a}^{a} h_{\pm}(x_1)\,u_{1,1}^{(0)}(x_1, \pm 0)^2\,dx_1 + O(\varepsilon^2) \tag{3.9}$$

where ν and E are Poisson's ratio and Young's modulus, respectively.

Since $4\mu(\mu+\lambda)(2\mu+\lambda)^{-1}u_{1,1}^{(0)} = \sigma_{11}(u^{(0)})$ as the edges of the crack M, we finally obtain the expression

$$\Pi(\varepsilon) = \Pi_0 - \varepsilon\frac{1-\nu^2}{2E}\sum_{\pm}\int_{-a}^{a} h_{\pm}(x_1)\,\sigma_{11}(u^{(0)}; x_1, \pm 0)^2\,dx_1 + O(\varepsilon^2) \tag{3.10}$$

4. Examples. 1°. Let us investigate the uniaxial extension of a plane with a thin slit G_ε described in Sect.1. We shall assume here that the crack edges are load-free, i.e. $\mathbf{q}^{\pm} = 0$ in (1.2). The boundary conditions (1.3) should be interpreted as follows:

$$\sigma_{11}(u;\ x) \to p\cos^2\beta;\ \sigma_{12}(u;\ x) \to p\sin\beta\cos\beta$$
$$\sigma_{22}(u;\ x) \to p\sin^2\beta \text{ as } |x| \to \infty$$

where p is the load intensity and β is the angle of inclination.

The stresses constructed according to the solution $\mathbf{u}^{(0)}$ of problem (1.4)-(1.6) are given, in this case, by the formulas (see e.g. /6/)

$$\sigma_{11}(u^{(0)};\ x) = p\{\cos^2\beta + \sin^2\beta[\operatorname{Re}Z(z) - x_2\operatorname{Im}Z'(z)] + \cos\beta\sin\beta[2\operatorname{Im}Z(z) + x_2\operatorname{Re}Z'(z)]\} \tag{4.1}$$
$$\sigma_{22}(u^{(0)};\ x) = p\{\sin^2\beta + \sin^2\beta[\operatorname{Re}Z(z) + x_2\operatorname{Im}Z'(z)] - x_2\cos\beta \times \sin\beta\operatorname{Re}Z'(z)\}$$
$$\sigma_{21}(u^{(0)};\ x) = p\{\sin\beta\cos\beta - x_2\sin^2\beta\operatorname{Re}Z'(z) + \cos\beta\sin\beta \times [\operatorname{Re}Z(z) - x_2\operatorname{Im}Z'(z)]\}$$
$$Z(z) = z(z^2 - a^2)^{-1/2} - 1,\ z = x_1 + ix_2$$

Substituting (4.1) into the boundary conditions (2.6) we obtain, for the vector $\mathbf{u}^{(1)}$,

$$\sigma_{22}(u^{(1)};\ x_1,\ \pm 0) = 0 \tag{4.2}$$
$$\sigma_{21}(u^{(1)}; x_1, \pm 0) = \pm p\frac{\partial}{\delta x_1}\left(h_\pm(x_1)\left\{\cos 2\beta \mp \frac{\sin 2\beta}{\sqrt{a^2 - x_1^2}}x_1\right\}\right)$$

The traces of the vector functions $\zeta^{(j,+)}$ governed by the relations (1.9), (1.10) on $M_\pm$, are given by the relations (see /6, 13/)

$$\zeta_2^{(1,+)}(x_1, \pm 0) = \zeta_2^{(2,+)}(x_1, \pm 0) = \pm\sqrt{\frac{a + x_1}{a - x_1}}$$
$$\zeta_1^{(1,+)}(x_1, \pm 0) = \frac{\varkappa - 1}{\varkappa + 1};\quad \zeta_2^{(2,+)}(x_1, \pm 0) = -\frac{\varkappa - 1}{\varkappa + 1}$$

Then from (4.2) we obtain

$$K_{(1,1)}^+ = 0,\ K_{(2,1)}^+ = \frac{p}{\sqrt{\pi a}}\int_{-a}^{a}\left\{(h_+(x_1) - h_-(x_1))\cos 2\beta - H(x_1)\frac{x_1\sin 2\beta}{\sqrt{a^2 - x_1^2}}\right\}\frac{a\,dx_1}{(a - x_1)\sqrt{a^2 - x_1^2}}$$

where $H(x_1) = h_+(x_1) + h_-(x_1)$ is the reduced crack width. Therefore, when $j = 2$, formula (2.8) is specifically as follows:

$$K_2^+(\varepsilon) = \frac{p}{\sqrt{\pi a}}\left\{\pi a\sin\beta\cos\beta + \varepsilon\int_{-a}^{a}\left\{[h_+(x_1) - h_-(x_1)]\cos 2\beta - H(x_1)\frac{x_1\sin 2\beta}{\sqrt{a^2 - x_1^2}}\right\}\frac{a\,dx_1}{(a - x_1)\sqrt{a^2 - x_1^2}}\right\} + O(\varepsilon^3) \tag{4.3}$$

Since $K_{(1.1)}^+ = 0$, it follows that formula (2.8) is lacking in content when $j = 1$. Let us make it more accurate, taking into account the second term of the asymptotic expression for $\mathbf{u}$. From (2.7) it follows that the averaged values of the sttesses $\sigma_{12}(u^{(2)}; x_1, \pm 0)$ are zero on the segment $[-a, a]$. Moreover, $\zeta_1^{(1,+)}(x_1, \pm 0) = \text{const}$. This means that by virtue of formula (1.8) the first boundary condition in (2.7) makes no contribution towards the coefficient $K_{(1,2)}^+$.

Let us transform the second boundary condition. According to (2.7) and the equations of equilibrium, we have

$$\sigma_{22}(u^{(2)}; x_1, \pm 0) = \frac{\partial}{\partial x_1}\left({}^1/{}_2 h(x_1)^2\sigma_{12,2}(u^{(0)}; x_1, \pm 0) + h(x_1)\frac{\partial}{\partial x_1}(h(x_1)\sigma_{11}(u^{(0)}; x_1, \pm 0)\right) = \frac{\partial^2}{\partial x_1^2}({}^1/{}_2 h(x_1)^2\sigma_{11}(u^{(0)}; x_1; \pm 0))$$

Therefore

$$K_{(1,2)}^+ = \frac{1}{4\sqrt{\pi a}}\sum_{\pm}\int_{-a}^{a}\frac{\partial^2}{\partial x_1^2}(h(x_1)^2\sigma_{11}(u^{(0)}; x_1, \pm 0)) \times \sqrt{\frac{a + x_1}{a - x_1}}dx_1 = \frac{1}{4\sqrt{\pi a}}\sum_{\pm}\int_{-a}^{a}h(x_1)^2 \times \sigma_{11}(u^{(0)}; x_1, \pm 0)\frac{\partial^2}{\partial x_1^2}\left(\sqrt{\frac{a + x_1}{a - x_1}}\right)dx_1 =$$
$$-\frac{p}{4}\sqrt{\frac{a}{\pi}}\int_{-a}^{a}\left\{\sum_{\pm}h_\pm(x_1)^2\cos 2\beta - [h_+(x_1)^2 - h_-(x_1)^2]\sin 2\beta\frac{x_1}{\sqrt{a^2 - x_1^2}}\right\}\frac{a + 2x_1}{a - x_1}\frac{dx_1}{(a^2 - x_1^2)^{3/2}}$$

Finally, when $j=1$, formula (2.8) becomes

$$K_1^+(\varepsilon) = p\sqrt{\frac{a}{\pi}}\left\{\pi\sin^2\beta - \frac{\varepsilon^2}{4}\int_{-a}^{a}\left\{\sum_{\pm} h_{\pm}(x_1)^2\cos 2\beta - \right.\right. \tag{4.4}$$

$$\left.\left.[h_+(x_1)^2 - h_-(x_1)^2]\frac{x_1\sin 2\beta}{\sqrt{a^2-x_1^2}}\right\}\frac{a+2x_1}{a-x_1}\frac{dx_1}{(a^2-x_1^2)^{1/2}} + O(\varepsilon^3)\right.$$

Let us discuss expressions (4.3) and (4.4) obtained for the stress intensity factors in the problem of uniaxial loading of a plane with a crack G_ε. We shall consider the case of transverse extension ($p=P>0$, $\beta=\pi/2$) and longitudinal compression ($p=-P<0$, $\beta=0$). By virtue of (4.3) and (4.4) we have

$$K_1^+(\varepsilon) - K_{(1,0)}^+ = \frac{P\varepsilon^2}{4}\sqrt{\frac{a}{\pi}}\int_{-a}^{a}\sum_{\pm} h_{\pm}(x_1)^2\frac{a+2x_1}{a-x_1}\frac{dx_1}{(a^2-x_1^2)^{1/2}} + O(\varepsilon^3)$$

(in the first case $K_{(1,0)}^+ = P\sqrt{\pi a}$, and in the second case $K_{(1,0)}^+=0$). In both cases the coefficient $K_2^+(\varepsilon)$ is equal to zero. In the case of longitudinal compression the coefficient $K_1^+(\varepsilon)$ is not, generally speaking, sign-constant. However, when the contour is symmetric (the functions $h_\pm(x_1)$ are even), the coefficient is positive. Moreover, even when the contour is not symmetric, the sum $K_1^+(\varepsilon)+K_1^-(\varepsilon)$ of the intensity coefficients at both tips of the crack is greater than zero. Actually, similar assertions also hold for the addition $K_1^+(\varepsilon)-K_{(1,0)}^+$ in the case of transverse extension.

2°. Let us consider the problem of the uniaxial compression of a bounded region Ω_ε introduced in Sect.1, in a direction parallel to the crack axis. In this case we have $\mathbf{p}^\pm=0$ and $\mathbf{p}(x) = -(n_1, 0)\,p$, in the boundary conditions (1.2) and (1.3) where p is the intensity of the compressive load. Therefore the displacements are given by the formula

$$u^\circ = p\,[4\mu(\mu+\lambda)]^{-1}(-(2\mu+\lambda)\,x_1,\ \lambda x_2) \tag{4.5}$$

Substituting (4.5) into (3.9) or (3.10), we obtain

$$\Delta\Pi = \Pi(\varepsilon) - \Pi_0 = -\varepsilon\frac{1-\nu^2}{2E}p^2\sum_{\pm}\int_{-a}^{a} h_\pm(x_1)\,dx_1 + O(\varepsilon^2) = -p^2\frac{(1-\nu^2)}{2E}S_\varepsilon + O(\varepsilon^2) \tag{4.6}$$

where S_ε is the area of the crack G_ε. Thus the increase in the potential energy when a crack appears within the body, parallel to the direction of the compressive load, is largely proportional to the area of the crack.

3°. Let us now apply the results of Sect.3 to the problem of uniaxial extension (compression) of a plane with a crack G_ε discussed in Sect.1. The potential energy of this body is infinite, therefore following /14/ we shall discuss the potential energy increment $\Delta\Pi$ accompanying the appearance of a crack G_ε in the plane. We shall calculate $\Delta\Pi$ using an example given in /15/. Let Q_D be a region containing a circle with centre at the origin of coordinates, with the parameter D fairly large. We denote by $\mathbf{u}(D,\varepsilon,x)$ the displacements caused by the uniaxial loading of the region $Q_D\setminus G_\varepsilon$. The loads are of intensity p, and are inclined at an angle β to the crack axis. In addition let $\mathbf{v}(x)$ be a solution of the analogous problem in a region without a crack

$$\mathbf{v}(x) = (2\mu)^{-1}p\,((\cos^2\beta - {}^1/_2\,\lambda(\lambda+\mu)^{-1})\ \ x_1 + \cos\beta\sin\beta\,x_2, \tag{4.7}$$
$$x_1\cos\beta\sin\beta + (\sin^2\beta - {}^1/_2\,\lambda(\lambda+\mu)^{-1})\,x_2)$$

and $\mathbf{w}(\varepsilon,x)$ be the solution of problem (1.1), (1.2) in the region $R^2\setminus G_\varepsilon$, vanishing at infinity

when $\mathbf{p}^\pm(\varepsilon, x) = (1+\varepsilon^2 h_\pm'(x_1)^2)^{-1/2}\mathbf{q}^\pm(\varepsilon, x)$,

where $\mathbf{q}^\pm(\varepsilon,x) = -p\{\varepsilon h_\pm'(x_1)\cos\beta \mp \sin\beta\}(\cos\beta,\ \sin\beta)$. We denote the potential energies corresponding to the problems in $Q_D\setminus G_\varepsilon$ and Q_D by Π_D and Π_D^0. Just as in /15/, we find from the results of /12/ that

$$\mathbf{u}(D,\varepsilon,x) = \mathbf{v}(x) + \mathbf{w}(\varepsilon,x) + O(D^{-1})$$
$$\Pi_D - \Pi_D^\circ = -\frac{1}{2}\left\{\int_{G_\varepsilon}\sigma_{ij}(v,x)\,\varepsilon_{ij}(v;x)\,dx + \int_{R^2\setminus G_\varepsilon}\sigma_{ij}(w;\varepsilon,x)\,\varepsilon_{ij}(w;\varepsilon,x)\,dx\right\} + o(1)$$

as $D\to+\infty$, where ε_{ij} are the deformation tensor components. Therefore, we have

$$\Delta\Pi = \lim_{D\to+\infty}(\Pi_D - \Pi_D^\circ) = -\frac{1}{2}\left\{\int_{G_\varepsilon}\sigma_{ij}(v;x)\,\varepsilon_{ij}(v;x)\,dx + \int_{R^2\setminus G_\varepsilon}\sigma_{ij}(w;\varepsilon,x)\,\varepsilon_{ij}(w;\varepsilon,x)\,dx\right\} \tag{4.8}$$

Let us compute the integrals on the right-hand side of (4.8). From (4.6) we obtain

$$\int_{G_\varepsilon} \sigma_{ij}(v;x)\,\varepsilon_{ij}(v;x)\,dx = p^2\,\frac{2\mu+\lambda}{4\mu(\lambda+\mu)}\,S_\varepsilon \tag{4.9}$$

We find the asymptotic form of the second integral in (4.7) using formulas (3.5), (3.8). The solution $\mathbf{w}^{(0)}(\varepsilon, x)$ of the limit problem (1.4), (1.5) in $R^2\setminus M$ can be written in the form

$$\mathbf{w}^{(0)}(\varepsilon, x) = \mathbf{z}^{(0)}(x) + \varepsilon\mathbf{z}^{(1)}(x)$$

where $\mathbf{z}^{(0)}$ and $\mathbf{z}^{(1)}$ are the solutions of the same problem with the right-hand sides

$$\mathbf{q}^{\pm,0}(x_1) = \pm p\sin\beta\,(\cos\beta,\ \sin\beta)$$

$$\mathbf{q}^{\pm,1}(x_1) = -ph_\pm'(x_1)\cos\beta\,(\cos\beta,\ \sin\beta)$$

Therefore we have

$$-\frac{1}{2}\int_{R^2\setminus G_\varepsilon} \sigma_{ij}(w;\varepsilon,x)\,\varepsilon_{ij}(w;\varepsilon,x)\,dx = -\frac{1}{2}\int_{\partial G_\varepsilon} \sigma^{(n)}(w;\varepsilon,x)\,w(\varepsilon,x)\,ds = \tag{4.10}$$

$$-\frac{1}{2}\sum_{\pm}\int_{-a}^{a}\{\mp\sigma^{(2)}(w^{(0)};\varepsilon,x_1,\pm 0)\,w^{(0)}(x_1,\pm 0)+$$

$$\varepsilon h_\pm(x_1)\{(2\mu+\lambda)\,[w_{1,1}^{(0)}(\varepsilon,x_1,\pm 0)^2 - w_{2,2}^{(0)}(\varepsilon,x_1,\pm 0)^2]+$$

$$\mu\,[w_{2,1}^{(0)}(\varepsilon,x_1,\pm 0)^2 - w_{1,2}^{(0)}(\varepsilon,x_1,\pm 0)^2]\}\}\,dx_1 + O(\varepsilon^2) =$$

$$-\frac{\varepsilon}{2}\sum_{\pm}\int_{-a}^{a} h_\pm(x_1)\,\{(2\mu+\lambda)\,[z_{1,1}^{(0)}(x_1,\pm 0)^2 - z_{2,2}^{(0)}(x_1,\pm 0)^2]+$$

$$\mu\,[z_{2,1}^{(0)}(x_1,\pm 0)^2 - z_{1,2}^{(0)}(x_1,\pm 0)^2]\}\,dx_1 -$$

$$\frac{1}{2}\int_{R^2\setminus M} \sigma_{ij}(z^{(0)}+\varepsilon z^{(1)};x)\,\varepsilon_{ij}(z^{(0)}+\varepsilon z^{(1)};x)\,dx + O(\varepsilon^2)$$

The last integral is equal to

$$-\frac{1}{2}\int_{R^2\setminus M}[\sigma_{ij}(z^{(0)};x)\,\varepsilon_{ij}(z^{(0)};x) + 2\varepsilon\sigma_{ij}(z^{(0)};x)\,\varepsilon_{ij}(z^{(1)};x)]\,dx + O(\varepsilon^2) = \tag{4.11}$$

$$-\frac{1}{2}\sum_{\pm}\mp\int_{-a}^{a}[\sigma^{(2)}(z^{(0)};x_1,\pm 0) + 2\varepsilon\sigma^{(2)}(z^{(1)};x_1,\pm 0)]\times$$

$$z^{(0)}(x_1,\pm 0)\,dx_1 + O(\varepsilon^2)$$

The traces of the displacement vector $\mathbf{z}^{(0)}$ at the edges $M^\pm$ of the cut M are given by the equations

$$\mathbf{z}^{(0)}(x_1,\ \pm 0) = \pm p\mu^{-1}(1-\nu)\sin\beta\,(a^2 - x_1^2)^{1/2}\,(\cos\beta,\ \sin\beta)$$

therefore formulas (4.8)-(4.11) yield the final expression for the potential energy increment

$$\Delta\Pi = -\pi a^2p^2\sin^2\beta\,\frac{1-\nu}{2\mu} - \frac{p^2(2\mu+\lambda)}{8\mu(\lambda+\mu)}\Big\{S_\varepsilon\,(1+\sin^2\beta\sin^2 2\beta) -$$

$$\varepsilon a^2\sin^2\beta\sin^2 2\beta\int_{-a}^{a}\frac{H(x_1)\,dx_1}{a^2-x_1^2}\Big\} + O(\varepsilon^2)$$

Thus in the case of longitudinal compression the increase in potential energy accompanying the appearance of a slit G_ε in the plane, is equal to

$$-p^2(2\mu+\lambda)\,[8\mu(\lambda+\mu)]^{-1}S_\varepsilon + O(\varepsilon^2)$$

which is, as expected, the same as that in formula (4.6).

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STUDY OF CRACK OPENING USING THE WEIGHTING FUNCTIONS METHOD*

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Some results of calculations of the opening of rectilinear, disc-like cracks under the action of a given system of forces, are given in /1-3/. A study of the opening of internal and surface cracks of more complex form is of interest, since in a number of cases it enables one to determine the depth of the crack from its known opening at the surface.

Formulas are obtained for the opening of elliptical, internal or surface cracks which occur when the body is acted upon by an arbitrary static load symmetrical about the plane of the crack.

1. Let us consider an elastic body with a rectilinear skew crack $0 \leqslant x \leqslant l$, internal or emerging at the surface $x = 0$. A weighting functions (WF) method was proposed in /4/ for computing the stress intensity factors (SIF) at the crack tip, and the possibility of using the method to determine the displacement field was suggested. When the elastic deformation energy $W(l)$ and the displacement of the upper edge of the crack $v(x, l)$ are both known for a certain external load, the WF can be found using the formula /4/

$$h(x, l) = \frac{1}{2}\left(\frac{1}{E'}\frac{\partial W}{\partial l}\right)^{-1/2}\frac{\partial v(x, l)}{\partial l} \tag{1.1}$$

where $E' = E/(1-\nu^2)$ for plane deformation, $E' = E$ for the state of plane stress, E is the modulus of elasticity, ν is Poisson's ratio and $h(x, l)$ is independent of the type of loading.

We have the following formula for the SIF $K(l)$ at the tip $x = l$:

$$K(l) = 2\int_0^l \sigma(x) h(x, l)\, dx \tag{1.2}$$

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